

STABILITY OF RIEMANNIAN MANIFOLDS WITH KILLING SPINORS

CHANGLIANG WANG

ABSTRACT. Riemannian manifolds with non-zero Killing spinors are Einstein manifolds. We prove that all complete Riemannian manifolds with imaginary Killing spinors are strictly stable by using a Bochner type formula in [DWW05] and [Wan91]. This stability result was also proved by Klaus Kröncke recently in a different way. A similar argument for real Killing spinors gives a stability condition for Riemannian manifold with real Killing spinors in term of a twisted Dirac operator. Existence of real Killing spinors is closely related to the Sasaki-Einstein structure. A regular Sasaki-Einstein manifold is essentially the total space of a certain principal S^1 -bundle over a Kähler-Einstein manifold. We prove that if the base space is a product of two Kähler-Einstein manifolds then the regular Sasaki-Einstein manifold is unstable.

1. INTRODUCTION

Let (M^n, g) be a Riemannian manifold with a non-zero Killing spinor σ with the Killing constant $\mu \neq 0$, i.e.

$$(1.1) \quad \nabla_X^S \sigma = \mu X \cdot \sigma,$$

for any vector field X , where ∇^S denotes the canonical connection on the spinor bundle induced by the Levi-Civita connection on the tangent bundle TM , and “ \cdot ” denotes the Clifford multiplication. Then the Riemannian manifold (M^n, g) is an Einstein manifold with scalar curvature $R = 4n(n-1)\mu^2$ (see, e.g. [Fri00]). Because the scalar curvature is real, μ can only be real or purely imaginary. A non-zero Killing spinor is said to be imaginary (resp. real) if its Killing constant is imaginary (resp. real).

Let (M^n, g) be an Einstein manifold. The operator $\nabla^* \nabla - 2\mathring{R}$ acting on symmetric 2-tensors $C^\infty(S^2(M))$ is called Einstein operator, where $S^2(M)$ is the bundle of symmetric 2-tensors and $(\mathring{R}h)_{ij} = R_{ikjl}h^{kl}$. Einstein metrics on a compact manifold M are critical points of total scalar curvature functional with the fixed volume 1. The second variation of the total scalar curvature functional with the fixed volume 1 at an Einstein metric g is given by $-\frac{1}{2}\langle \nabla^* \nabla h - 2\mathring{R}h, h \rangle_{L^2(M)}$, when restricted in traceless transverse direction, i.e. $h \in C^\infty(S^2(M))$ satisfying $tr_g h = 0$ and $\delta_g h = 0$, where $\delta_g h$ is the divergence of h . We say an Einstein manifold (M^n, g) is stable if $\langle \nabla^* \nabla h - 2\mathring{R}h, h \rangle_{L^2(M)} \geq 0$ for all traceless transverse symmetric 2-tensors h , and otherwise, (M^n, g) is unstable. (M^n, g) is strictly

stable if $\langle \nabla^* \nabla h - 2\mathring{R}h, h \rangle_{L^2(M)} \geq c \langle h, h \rangle_{L^2(M)}$ for some constant $c > 0$. If the manifold is non-compact, we only consider compactly supported symmetric 2-tensors h .

The stability problem of Einstein metrics was also similarly studied with respect to variation formulae of Perelman's ν -entropy (see, e.g. [Per02] and [CZ12]) for Einstein metrics with positive Ricci curvature, and also variation formulae of ν_+ -entropy (see, e.g. [FIN05] and [Zhu11]) for Einstein metrics with negative Ricci curvature. For example, H-D. Cao and C. He studied stability of Einstein metrics with respect to ν -entropy on symmetric spaces of compact type in [CH13].

In this paper, we will study the stability of complete Riemannian manifolds with non-zero Killing spinors, which then are Einstein manifolds and have been classified in [Bär93], [Bau89a], and [Bau89b]. Riemannian manifolds with real and imaginary Killing spinors have several very distinct properties. For example, Riemannian manifolds with non-zero real Killing spinors are compact. On the other hand, Riemannian manifolds with non-zero imaginary Killing spinors are non-compact (see [CGLS86] and [Bau89b]). So we study these two kinds of manifolds separately.

If we set $\mu = 0$ in (1.1), i.e. $\nabla_X^S \sigma = 0$ for any vector field X , then σ is called a parallel spinor. Riemannian manifolds with non-zero parallel spinors are Ricci-flat, i.e. Ricci curvature is zero. X. Dai, X. Wang, and G. Wei proved that manifolds with non-zero parallel spinors are stable in [DWW05] by deriving a Bochner type formula, and rediscovering a result in [Wan91], also see [GHP03] for the formula.

Moreover, an imaginary Killing spinor is of type I if there exists a vector field X such that $X \cdot \sigma = \sqrt{-1}\sigma$, and otherwise, σ is of type II. H. Baum proved that n -dimensional complete Riemannian manifolds with imaginary Killing spinors of type II with Killing constant $\sqrt{-1}\nu$ are isometric to the n -dimensional hyperbolic space $H_{-4\nu^2}^n$ with constant sectional curvature $-4\nu^2$. N. Koiso proved that Einstein manifolds with negative sectional curvature, in particular, hyperbolic spaces, are stable in [Koi79] (also see [Bes87]). Indeed, by the first inequality in 12.70 in [Bes87], one can see that $\langle \nabla^* \nabla h - 2\mathring{R}h, h \rangle_{L^2} \geq 4(n-2)\nu^2 \langle h, h \rangle_{L^2}$ for all compactly supported traceless transverse 2-tensors h on the hyperbolic space $H_{-4\nu^2}^n$.

Therefore, we focus on Riemannian manifolds with imaginary Killing spinors of type I and ones with real Killing spinors in this paper. Recently, in [Krö15], K. Kröncke proved that complete Riemannian manifolds with non-zero imaginary Killing spinors are stable by using a warped product structure of these manifolds and the stability result in [DWW05]. We obtain an estimate for Einstein operator on complete Riemannian manifolds with imaginary Killing spinors of type I by using a Bochner type formula in [DWW05] and [Wan91], and meanwhile, provide a shorter proof for this stability result.

Theorem 1.1. *Let (M^n, g) be a complete Riemannian manifold with a non-zero imaginary Killing spinor of type I with the imaginary Killing constant μ . We have*

$$(1.2) \quad \int_M \langle \nabla^* \nabla h - 2\mathring{R}h, h \rangle d\text{vol}_g \geq -[2(n-2) - 4]\mu^2 \int_M \langle h, h \rangle d\text{vol}_g.$$

for all compactly supported traceless transverse symmetric 2-tensor h .

Corollary 1.2. *Complete Riemannian manifolds with non-zero imaginary Killing spinors are strictly stable.*

In the case of real Killing spinors, we have the following estimate.

Theorem 1.3. ([GHP03], [Wan91]) *Let (M^n, g) be a Riemannian manifold with non-zero real Killing spinor with Killing constant μ , then for all traceless transverse $h \in C^\infty(S^2(M))$,*

$$(1.3) \quad \begin{aligned} & \int_M \langle \nabla^* \nabla h - 2\mathring{R}h, h \rangle d\text{vol}_g \\ &= \int_M \langle D\Phi(h), D\Phi(h) \rangle d\text{vol}_g - 2\mu \int_M \langle D\Phi(h), \Phi(h) \rangle d\text{vol}_g \\ & \quad - n(n-2)\mu^2 \int_M \langle h, h \rangle d\text{vol}_g. \end{aligned}$$

Unlike the case of imaginary Killing spinors, from this estimate we cannot conclude a general stability result. Actually, we have both stable and unstable examples: standard spheres are stable Riemannian manifolds with real Killing spinors; the Jensen's sphere (also called the squashed sphere) is an unstable Riemannian manifold with real Killing spinor. We refer to [ADP83], [Bes87], [Jen73], and [Spa11] for details about the Jensen's sphere. We obtain a stability condition for manifolds with non-zero real Killing spinors from Theorem 1.3.

Corollary 1.4. *The Riemannian manifold with non-trivial real Killing spinor with Killing constant μ is stable if the twisted Dirac operator D satisfies*

$$(D - \mu)^2 \geq (n-1)^2 \mu^2,$$

on $\{\Phi(h) : h \in C^\infty(S^2(M)), \text{tr} h = 0, \delta h = 0\}$.

Most Riemannian manifolds with non-zero real Killing spinors are either standard spheres in even dimensions, or Sasaki-Einstein in odd dimensions. And all regular Sasaki-Einstein manifolds are the total spaces of principal S^1 -bundles over Kähler-Einstein manifolds. Let $\pi : (M^{2p+1}, g) \rightarrow (B^{2p}, G, J)$ be a principal S^1 -bundle with a connection η , where (M^{2p+1}, g) is regularly Sasaki-Einstein, (B^{2p}, G, J) is Kähler-Einstein, and π is a Riemannian submersion. Here G is the Kähler metric on B^{2p} , and J is the almost complex structure on B^{2p} . In the following, $\tilde{h} = \pi^* h$, for all $h \in C^\infty(S^2(B))$.

Proposition 1.5.

$$(1.4) \quad \langle (\nabla^g)^* \nabla^g \tilde{h} - 2\mathring{R}^g \tilde{h}, \tilde{h} \rangle = \langle (\nabla^G)^* \nabla^G h - 2\mathring{R}^g h, h \rangle + 4\langle h, h \rangle + 4\langle h \circ J, h \rangle \circ \pi,$$

and therefore,

$$(1.5) \quad \begin{aligned} & \int_M \langle (\nabla^g)^* \nabla^g \tilde{h} - 2\mathring{R}^g \tilde{h}, \tilde{h} \rangle d\text{vol}_g \\ &= \int_B \langle (\nabla^G)^* \nabla^G h - 2\mathring{R}^g h, h \rangle + 4\langle h, h \rangle + 4\langle h \circ J, h \rangle d\text{vol}_G, \end{aligned}$$

where $h \circ J \in C^\infty(S^2(B))$ with $h \circ J(X, Y) = h(JX, JY)$.

Corollary 1.6. *If there exists a traceless transverse symmetric 2-tensor $h \in C^\infty(S^2 B)$ such that $\int_B \langle (\nabla^G)^* \nabla^G h - 2\mathring{R}^g h, h \rangle d\text{vol}_G < -8 \int_B \langle h, h \rangle d\text{vol}_G$, then (M^{2p+1}, g) is unstable.*

Corollary 1.7. *If the base space (B^{2p}, g) is a product of two Kähler-Einstein manifolds, then (M^{2p+1}, g) is unstable.*

2. RIEMANNIAN MANIFOLDS WITH IMAGINARY KILLING SPINORS

In this section, we review classification results of Riemannian manifolds with Killing spinors and some properties of Killing spinors. We will mainly focus on complete Riemannian manifolds with imaginary Killing spinors studied in [Bau89a] and [Bau89b], because Baum's results about the structure of complete Riemannian manifolds with imaginary Killing spinors play a very important role in our estimate of the Einstein operator on these manifolds.

Let us first recall two differences between manifolds with real Killing spinors and manifolds with imaginary Killing spinors pointed out in [Bau89b] (also see [CGLS86]):

- (1) Let (M^n, g) be a complete Riemannian manifold with a Killing spinor σ . If σ is real with non-zero real Killing constant, then M^n is compact. If σ is imaginary, then M^n is non-compact.
- (2) Let $f(x) := \langle \sigma(x), \sigma(x) \rangle_{s_x}$ denote the length function of a non-zero Killing spinor σ . If σ is real, then f is constant. If σ is imaginary, then f is a non-constant and nowhere vanishing function.

As pointed out by Klaus Kröncke in [Krö15], the fact that the length function f of an imaginary Killing spinor is not constant will cause some issues when we use the Bochner type argument in [DWW05] to estimate the Einstein operator on a Riemannian manifold with imaginary Killing spinors. In order to deal with the issues, we investigate the length function f more carefully, and we recall some properties of the length function f proved in [Bau89b]. Let (M^n, g) be a complete Riemannian manifold with an imaginary Killing spinor σ with Killing constant $\mu = \sqrt{-1}\nu$.

Lemma 2.1. ([Bau89b])

- (1) *The function*

$$(2.1) \quad q_\sigma(x) := f^2(x) - \frac{1}{4\nu^2} |\nabla f(x)|^2$$

is constant on M^n .

(2) Let $\{e_1, \dots, e_n\}$ be a local o.n. frame of TM around x . Then we have

$$(2.2) \quad \operatorname{Re}\langle e_i \cdot \sigma(x), e_j \cdot \sigma(x) \rangle = \delta_{ij} f(x),$$

where Re means taking the real part.

(3) Let dist denote the distance in \mathcal{S}_x with respect to the real scalar product $\operatorname{Re}\langle \cdot, \cdot \rangle_{\mathcal{S}_x}$. Then

$$(2.3) \quad q_\sigma = f(x) \cdot \operatorname{dist}^2(V_\sigma, \sqrt{-1}\sigma(x)) \geq 0,$$

where $V_\sigma(x) = \{X \cdot \sigma(x) \mid X \in T_x M\} \subset \mathcal{S}_x$.

As in [Bau89b], σ a Killing spinor is of type I if $q_\sigma = 0$ and a Killing spinor is of type II if $q_\sigma > 0$. By (2.3), this is essentially the same as the simple characteristic of Killing spinors of type I and II mentioned in Introduction. H. Baum has the following classification results for complete Riemannian manifold with imaginary Killing spinors.

Theorem 2.2. ([Bau89b]) *Let (M^n, g) be a complete connected Riemannian manifold with an imaginary Killing spinor of type II with the Killing constant $\sqrt{-1}\nu$. Then (M^n, g) is isometric to the hyperbolic space $H_{-4\nu^2}^n$ with the constant sectional curvature $-4\nu^2$.*

Theorem 2.3. ([Bau89a][Bau89b]) *Let (M^n, g) be a complete connected Riemannian manifold with an imaginary Killing spinor of type I with the Killing constant $\sqrt{-1}\nu$. Then (M^n, g) is isometric to a warped product $(F^{n-1} \times \mathbb{R}, e^{-4\nu t}h + dt^2)$, where (F^{n-1}, h) is a complete Riemannian manifold with a non-zero parallel spinor.*

Conversely, let (F^{n-1}, h) be a complete Riemannian manifold with non-zero parallel spinors, then the warped product $(M^n, g) := (F^{n-1} \times \mathbb{R}, e^{-4\nu t}h + dt^2)$ is a complete Riemannian manifold with imaginary Killing spinors of type I.

Recall how to construct a Killing spinor of type I on $(F^{n-1} \times \mathbb{R}, e^{-4\nu t}h + dt^2)$ from a parallel spinor on (F^{n-1}, h) . When $n-1$ is even, the spinor bundle over the warped product $(F^{n-1} \times \mathbb{R}, e^{-4\nu t}h + dt^2)$ is isometric to the tensor product of the spinor bundle over (F^{n-1}, h) and the spinor bundle over (\mathbb{R}, dt^2) . When $n-1$ is odd, the spinor bundle over $(F^{n-1} \times \mathbb{R}, e^{-4\nu t}h + dt^2)$ is isometric to the direct sum of two copies of the tensor product of the spinor bundle over (F^{n-1}, h) and the spinor bundle over (\mathbb{R}, dt^2) . The spinor bundle over (\mathbb{R}, dt^2) is a trivial 1-dimensional complex vector bundle. We will use the same notation to denote two isometric spinors.

- If $n-1$ is even, and parallel spinor on F^{n-1} is $\psi = (\psi^+, \psi^-)$, where the decomposition is the $\sqrt{-1}$ and $-\sqrt{-1}$ eigenspaces decomposition for the action of the complex volume $\omega_{\mathbb{C}} = (\sqrt{-1})^{\frac{n}{2}} e_1 \cdots e_{n-1}$ on the spinor bundle on F^{n-1} , then we can take

$$(2.4) \quad \sigma = e^{-\nu t} \psi^+ \otimes 1$$

as an imaginary Killing spinor of type I on the warped product manifold.

- If $n - 1$ is odd, and parallel spinor on F^{n-1} is ψ , then we can take

$$(2.5) \quad \sigma = e^{-\nu t}(\psi \otimes 1, \hat{\psi} \otimes 1)$$

as a Killing spinor of type I on the warped product manifold, where “ $\hat{\cdot}$ ” denotes the isomorphism between two spin representations coming from projections to the first and the second components of $Cl(n-1) \otimes \mathbb{C} = End(\mathbb{C}^{\frac{n-2}{2}}) \oplus End(\mathbb{C}^{\frac{n-2}{2}})$.

Because the length of a parallel spinor is constant, we can always normalize the parallel spinor ψ on F so that for the Killing spinor σ in (2.4) and (2.5) we have

$$\langle \sigma, \sigma \rangle = e^{-2\nu t}.$$

Thus for the Killing spinor obtained above we have the length function

$$(2.6) \quad f = e^{-2\nu t}$$

only depending on the t variable on \mathbb{R} factor. We can also see that $q_\sigma = 0$. Moreover, we can see that the action of the vector field $\frac{\partial}{\partial t}$ on the Killing spinor σ is given by

$$(2.7) \quad \left(\frac{\partial}{\partial t}\right) \cdot \sigma = \sqrt{-1}\sigma.$$

3. BOCHNER TYPE FORMULA

In this section, we recall a Bochner type formula coming from Killing spinors in [DWW05] and [Wan91] and present a proof.

Let (M^n, g) be a Riemannian spin manifold with spinor bundle $\mathcal{S} \rightarrow M$. The curvature of a connection ∇ on a vector bundle $E \rightarrow M$ is defined as

$$(3.1) \quad R_{XY}\sigma = -\nabla_X \nabla_Y \sigma + \nabla_Y \nabla_X \sigma + \nabla_{[X,Y]}\sigma,$$

for a section $\sigma \in C^\infty(E)$ and vector field $X, Y \in C^\infty(TM)$. Let R^S be the curvature of ∇^S on the spinor bundle. Let $\{e_1, \dots, e_n\}$ be a local orthonormal frame of the tangent bundle and $\{e^1, \dots, e^n\}$ be its dual frame. We have

$$(3.2) \quad R_{XY}^S \sigma = \frac{1}{4} R(X, Y, e_i, e_j) e_i e_j \cdot \sigma,$$

for any spinor σ . If there exists a Killing spinor σ with Killing constant μ , the Ricci curvature tensor satisfies

$$(3.3) \quad R_{ij} = 4\mu^2(n-1)g_{ij},$$

(see, e.g. [Fri00]). As in [DWW05], we define a linear map $\Phi : S^2(M) \rightarrow \mathcal{S} \otimes T^*M$ as

$$(3.4) \quad \Phi(h) = h_{ij} e_i \cdot \sigma \otimes e^j.$$

Proposition 3.1. ([Wan91]) *Let D be the twisted Dirac operator acting on $\mathcal{S} \otimes T^*M$, and h be a symmetric 2-tensor on M . Then*

$$(3.5) \quad \begin{aligned} D^*D\Phi(h) = & \Phi(\Delta_E h) + n(n-2)\mu^2\Phi(h) + 2\mu D\Phi(h) \\ & + 4\mu^2(\text{tr}h)e_j \cdot \sigma \otimes e^j - 4\mu(\delta h)_j \cdot \sigma \otimes e^j. \end{aligned}$$

Proof. Fix a point $x \in M$, choose a local orthonormal frame $\{e_1, \dots, e_n\}$ such that $\nabla e_i = 0$ at x . Then, at x ,

$$(3.6) \quad \begin{aligned} D^*D\Phi(h) &= \nabla_{e_k} \nabla_{e_l} h_{ij} e_k e_l e_i \cdot \sigma \otimes e^j + \nabla_{e_l} h_{ij} e_k e_l e_i \cdot \nabla_{e_k}^S \sigma \otimes e^j \\ &\quad + \nabla_{e_k} h_{ij} e_k e_l e_i \cdot \nabla_{e_l}^S \sigma \otimes e^j + h_{ij} e_k e_l e_i \cdot \nabla_{e_k}^S \nabla_{e_l}^S \sigma \otimes e^j \\ &= \nabla_{e_k} \nabla_{e_l} h_{ij} e_k e_l e_i \cdot \sigma \otimes e^j + \nabla_{e_l} h_{ij} (e_k e_l + e_l e_k) e_i \cdot \nabla_{e_k}^S \sigma \otimes e^j \\ &\quad + h_{ij} e_k e_l e_i \cdot \nabla_{e_k}^S \nabla_{e_l}^S \sigma \otimes e^j \\ &= \nabla_{e_k} \nabla_{e_l} h_{ij} e_k e_l e_i \cdot \sigma \otimes e^j - 2\mu \nabla_{e_k} h_{ij} e_i e_k \cdot \sigma \otimes e^j \\ &\quad + \mu^2 h_{ij} e_k e_l e_i e_l e_k \cdot \sigma \otimes e^j \\ &= -\nabla_{e_k} \nabla_{e_k} h_{ij} e_i \cdot \sigma \otimes e^j - \frac{1}{2} R_{e_k e_l} h_{ij} e_k e_l e_i \cdot \sigma \otimes e^j \\ &\quad - 2\mu \nabla_{e_k} h_{ij} e_i e_k \cdot \sigma \otimes e^j + (n-2)^2 \mu^2 h_{ij} e_i \cdot \sigma \otimes e^j \\ &= \Phi(\nabla^* \nabla h) + \frac{1}{2} R_{kljp} h_{ip} e_k e_l e_i \cdot \sigma \otimes e^j + \frac{1}{2} R_{klip} h_{pj} e_k e_l e_i \cdot \sigma \otimes e^j \\ &\quad - 2\mu \nabla_{e_k} h_{ij} e_i e_k \cdot \sigma \otimes e^j + (n-2)^2 \mu^2 \Phi(h). \end{aligned}$$

In the third equality, we use the Clifford relation $e_k e_l + e_l e_k = -2\delta_{kl}$, and $\nabla_X^S \sigma = \mu X \cdot \sigma$ for any vector field X . In the fourth equality, we use twice the fact

$$e_l e_i e_l \cdot \phi = (n-2)e_i \cdot \phi$$

for any spinor ϕ , which can easily be obtained by using the Clifford relation.

By using the Clifford relation, (3.2), and (3.3), we have

$$(3.7) \quad \frac{1}{2} R_{kljp} h_{ip} e_k e_l e_i \cdot \sigma \otimes e^j = \Phi(-2\mathring{R}h) - 4\mu^2 \Phi(h) + 4\mu^2 \text{tr} h e_j \cdot \sigma \otimes e^j,$$

$$(3.8) \quad \frac{1}{2} R_{klip} h_{pj} e_k e_l e_i \cdot \sigma \otimes e^j = 4(n-1)\mu^2 \Phi(h),$$

$$(3.9) \quad -2\mu \nabla_{e_k} h_{ij} e_i e_k \cdot \sigma \otimes e^j = -4\mu(\delta h)_j \sigma \otimes e^j + 2\mu e_k \cdot \Phi(\nabla_{e_k} h),$$

$$(3.10) \quad e_k \cdot \Phi(\nabla_{e_k} h) = D\Phi(h) - (n-2)\mu\Phi(h).$$

By plugging (3.7), (3.8), (3.9) and (3.10) into (3.6), we get (3.5). \square

4. STABILITY OF RIEMANNIAN MANIFOLDS WITH IMAGINARY KILLING SPINORS

In this section, we obtain an estimate for the Einstein operator on complete Riemannian manifolds with imaginary Killing spinors of type I. As a consequence of the estimate and Baum's classification results, we prove that all complete Riemannian manifolds with imaginary Killing spinors are strictly stable.

Let (M^n, g) be a Riemannian manifold with an imaginary Killing spinor σ of type I with the Killing constant $\mu = \sqrt{-1}\nu$. We have the following property for the map Φ defined in (3.4).

Lemma 4.1. *For all $h, \tilde{h} \in C^2(S^M)$, we have*

$$(4.1) \quad \operatorname{Re}\langle \Phi(h), \Phi(\tilde{h}) \rangle = \langle h, \tilde{h} \rangle f,$$

where $f = \langle \sigma, \sigma \rangle$ is the length function.

Proof.

$$\begin{aligned} \operatorname{Re}\langle \Phi(h), \Phi(\tilde{h}) \rangle &= \operatorname{Re}(h_{ij}\tilde{h}_{kl}\langle e_i \cdot \sigma \otimes e^j, e_k \cdot \sigma \otimes e^l \rangle) \\ &= \operatorname{Re}(h_{ij}\tilde{h}_{kj}\langle e_i \cdot \sigma, e_k \cdot \sigma \rangle) \\ &= h_{ij}\tilde{h}_{kj}\operatorname{Re}\langle e_i \cdot \sigma, e_k \cdot \sigma \rangle \\ &= h_{ij}\tilde{h}_{ij}f. \end{aligned}$$

In the last step, we use (2.2). □

Lemma 4.2. *If σ is a Killing spinor of type I as in (2.4) or (2.5), then we have*

$$(4.2) \quad \left\| \left(\frac{\partial}{\partial t} \right) \cdot \Phi(h) \right\| = \|\Phi(h)\|.$$

Proof. Choose a local o.n. frame of TM as $\{e_1 = \frac{\partial}{\partial r}, e_2, \dots, e_n\}$. Then by (2.7), we have

$$\begin{aligned} \left(\frac{\partial}{\partial t} \right) \cdot \Phi(h) &= \left(\frac{\partial}{\partial t} \right) \cdot (h_{1j} \left(\frac{\partial}{\partial t} \right) \cdot \sigma \otimes e^j + \sum_{i \geq 2} h_{ij} e_i \cdot \sigma \otimes e^j) \\ &= \sqrt{-1} h_{1j} \left(\frac{\partial}{\partial t} \right) \cdot \sigma \otimes e^j - \sqrt{-1} \sum_{i \geq 2} h_{ij} e_i \cdot \sigma \otimes e^j. \end{aligned}$$

Then by (2.2), we have

$$\begin{aligned}
\|(\frac{\partial}{\partial t}) \cdot \Phi(h)\|^2 &= \text{Re}\langle (\frac{\partial}{\partial t}) \cdot \Phi(h), (\frac{\partial}{\partial t}) \cdot \Phi(h) \rangle \\
&= \text{Re}\langle \sqrt{-1}h_{1j}(\frac{\partial}{\partial t}) \cdot \sigma \otimes e^j - \sqrt{-1}\sum_{i \geq 2} h_{ij}e_i \cdot \sigma \otimes e^j, \\
&\quad \sqrt{-1}h_{1l}(\frac{\partial}{\partial t}) \cdot \sigma \otimes e^l - \sqrt{-1}\sum_{k \geq 2} h_{kl}e_k \cdot \sigma \otimes e^l \rangle \\
&= h_{ij}h_{ij}f \\
&= \|\Phi(h)\|^2.
\end{aligned}$$

□

Theorem 4.3. *Let (M^n, g) be a complete Riemannian manifold with an imaginary Killing spinor σ of type I with Killing constant $\mu = \sqrt{-1}\nu$. Then we have*

$$(4.3) \quad \int_M \langle \Delta_E h, h \rangle d\text{vol}_g \geq [n(n-2) - 4]\nu^2 \int_M \langle h, h \rangle d\text{vol}_g,$$

for all compactly supported traceless transverse $h \in C_0^\infty(S^2(M))$.

Proof. By Proposition 3.1, for all traceless transverse symmetric 2-tensor h ,

$$(4.4) \quad \Phi(\Delta_E h) = D^*D\Phi(h) - n(n-2)\mu^2\Phi(h) - 2\mu D\Phi(h).$$

By Theorem 2.3, we can take a Killing spinor as in (2.4) or (2.5) depending on dimension n of the manifold. Then we know the length function is given by

$$(4.5) \quad f = e^{-2\nu t}.$$

By (4.4), and Lemma 4.1, for all traceless transverse $h \in C_0^\infty(S^2(M))$, we have

$$\begin{aligned}
&\int_M \langle \Delta_E h, h \rangle d\text{vol}_g \\
&= \int_M \frac{\text{Re}\langle \Phi(\Delta_E h), \Phi(h) \rangle}{f} d\text{vol}_g \\
(4.6) \quad &= \int_M \frac{\text{Re}\langle D^*D\Phi(h), \Phi(h) \rangle}{f} d\text{vol}_g - n(n-2)\mu^2 \int_M \frac{\langle \Phi(h), \Phi(h) \rangle}{f} d\text{vol}_g \\
&\quad + \int_M \frac{\text{Re}\langle -2\mu D\Phi(h), \Phi(h) \rangle}{f} d\text{vol}_g
\end{aligned}$$

By using (4.5) and doing an integration by parts, we obtain

$$\int_M \frac{\text{Re}\langle D^*D\Phi(h), \Phi(h) \rangle}{f} d\text{vol}_g = \int_M \frac{\|D\Phi(h)\|^2}{f} d\text{vol}_g + \int_M \frac{\text{Re}\langle D\Phi(h), 2\nu(\frac{\partial}{\partial t}) \cdot \Phi(h) \rangle}{f} d\text{vol}_g.$$

By Cauchy inequality, we have

$$\begin{aligned}
\operatorname{Re}\langle D\Phi(h), 2\nu(\frac{\partial}{\partial t}) \cdot \Phi(h) \rangle &\geq -\|D\Phi(h)\| \cdot \|2\nu(\frac{\partial}{\partial t}) \cdot \Phi(h)\| \\
&\geq -\frac{\|D\Phi(h)\|^2 + 4\nu^2\|(\frac{\partial}{\partial t}) \cdot \Phi(h)\|^2}{2} \\
&= -\frac{\|D\Phi(h)\|^2 + 4\nu^2\|\Phi(h)\|^2}{2}
\end{aligned}$$

Thus we have

$$(4.7) \quad \int_M \frac{\operatorname{Re}\langle D^*D\Phi(h), \Phi(h) \rangle}{f} d\operatorname{vol}_g \geq \frac{1}{2} \int_M \frac{\|D\Phi(h)\|^2}{f} d\operatorname{vol}_g - 2\nu^2 \int_M \langle h, h \rangle d\operatorname{vol}_g$$

Similarly, by Cauchy inequality, we have

$$(4.8) \quad \int_M \frac{\operatorname{Re}\langle -2\mu D\Phi(h), \Phi(h) \rangle}{f} d\operatorname{vol}_g \geq -\frac{1}{2} \int_M \frac{\|D\Phi(h)\|^2}{f} d\operatorname{vol}_g - 2\nu^2 \int_M \langle h, h \rangle d\operatorname{vol}_g$$

Plugging (4.7) and (4.8) into (4.6), we complete the proof. \square

Then Theorem 4.3 enables us to prove the following stability result recently obtained in [Krö15] in a differential way.

Corollary 4.4. *Complete Riemannian manifolds with non-zero imaginary Killing spinors are strictly stable.*

Proof. By Theorem 2.2, complete Riemannian manifolds with Killing spinors of type II are isometric to hyperbolic spaces, and therefore are strictly stable (see [Koi79], and the proof of Theorem 12.67 in [Bes87]). Let (M^n, g) be a Riemannian manifold with Killing spinors of type I. If $n \geq 4$, then by Theorem 4.3, (M^n, g) is strictly stable. If $n \leq 3$, we know it has negative constant sectional curvature, and therefore is also strictly stable. \square

5. STABILITY OF RIEMANNIAN MANIFOLDS WITH REAL KILLING SPINORS

In this section, we give a stability condition for manifolds with real Killing spinors in terms of a twisted Dirac operator. Because the length function of a real Killing spinor is constant, an estimate for the Einstein operator can be obtained easier than the case of imaginary Killing spinor. However, unlike imaginary Killing spinor case, from the estimate we cannot conclude a general stability result for manifolds with real Killing spinors.

Let (M^n, g) be a Riemannian manifold with a real Killing spinor σ with Killing constant μ . Without loss of generality, we can choose σ to be of unit length.

Lemma 5.1. *For all $h, \tilde{h} \in C^\infty(S^2(M))$, we have*

$$\operatorname{Re}\langle \Phi(h), \Phi(\tilde{h}) \rangle = \langle h, \tilde{h} \rangle.$$

Then by Proposition 3.1, Lemma 5.1, and the fact that $\mu \int_M \langle D\Phi(h), \Phi(h) \rangle d\operatorname{vol}_g$ is real, we obtain the following estimate for the Einstein operator $\nabla^* \nabla - 2\hat{R}$.

Theorem 5.2. ([GHP03], [Wan91]) *If the Killing constant μ is real, then, for all traceless transverse $h \in C^\infty(S^2(M))$,*

$$(5.1) \quad \int_M \langle \Delta_E h, h \rangle d\text{vol}_g = \int_M \langle D\Phi(h), D\Phi(h) \rangle d\text{vol}_g - 2\mu \int_M \langle D\Phi(h), \Phi(h) \rangle d\text{vol}_g \\ - n(n-2)\mu^2 \int_M \langle h, h \rangle d\text{vol}_g.$$

Remark 5.3. As mentioned in [Die13] and [Krö15], Theorem 5.2 has been used to obtain a lower bound on the eigenvalues of the Einstein operator in [GHP03]. The lower bound is $-(n-1)^2\mu^2$, as we can also see in the following Corollary 5.4.

Corollary 5.4. *The Riemannian manifold with non-zero real Killing spinor with Killing constant μ is stable if the twisted Dirac operator D satisfies*

$$(D - \mu)^2 \geq (n-1)^2\mu^2,$$

on $\{\Phi(h) : h \in C^\infty(S^2(M)), \text{tr} h = 0, \delta h = 0\}$.

Proof. By Theorem 5.2, for traceless transverse symmetric 2-tensor h , we have

$$(5.2) \quad \int_M \langle \Delta_E h, h \rangle d\text{vol}_g = \int_M \langle (D - \mu)^2 \Phi(h), \Phi(h) \rangle d\text{vol}_g \\ - (n-1)^2\mu^2 \int_M \langle h, h \rangle d\text{vol}_g.$$

This implies the stability condition. □

6. SOME UNSTABLE REGULAR SASAKI-EINSTEIN MANIFOLDS

In this section, we study instability of regular Sasaki-Einstein manifolds, which are essentially total spaces of principal circle bundles over Kähler-Einstein manifolds with positive first Chern classes. A product of two Einstein manifolds (B^{n_1}, g_1) and (B^{n_2}, g_2) with the same positive Einstein constant is an unstable Einstein manifold. Indeed, $h = \frac{g_1}{n_1} - \frac{g_2}{n_2}$ is an unstable traceless transverse direction. We show that if the base manifold of a regular Sasaki-Einstein manifold is a product of two Kähler-Einstein manifolds then we obtain an unstable direction on the Sasaki-Einstein manifold by lifting this unstable direction on the base Kähler-Einstein manifold to the total space.

Let us first recall some basic facts about Sasaki manifolds. For details, we refer to [Bla10] and [FOW09]. A quick definition of Sasaki manifold is given as the following, see, e.g. [FOW09].

Definition 6.1. (Definition 1 of Sasaki Manifolds) *(M^n, g) is said to be a Sasaki manifold if the cone $(\mathbb{R}_+ \times M, dr^2 + r^2 g)$ is Kähler, where $\mathbb{R}_+ = (0, +\infty)$, and r is coordinate on \mathbb{R}_+ .*

Remark 6.2. From Definition 6.1, we note that a Sasaki manifold has to be of odd dimension.

There are several equivalent definitions of Sasaki manifolds. The one given in the following looks more complicated and tells us more about structure on Sasaki manifolds themselves.

Definition 6.3. (Definition 2 of Sasaki manifolds) *Let $(M^{2p+1}, g, \phi, \eta, \xi)$ be a Riemannian manifold of odd dimension $2p + 1$ with a $(1, 1)$ -tensor ϕ , 1-form η , and a vector field ξ . It is a Sasaki manifold, if*

- (1) $\eta \wedge (d\eta)^p \neq 0$,
- (2) $\eta(\xi) = 1$,
- (3) $\phi^2 = -id + \eta \otimes \xi$,
- (4) $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$,
- (5) $g(X, \phi Y) = d\eta(X, Y)$,
- (6) *the almost complex structure on $M^{2p+1} \times \mathbb{R}$ defined by*

$$J(X, f \frac{d}{dr}) = (\phi X - f\xi, \eta(X) \frac{d}{dr})$$

is integrable,

for all vector fields X and Y on M^{2p+1} . The vector ξ is called Reeb vector field. And this is a regular Sasaki manifold if the Reeb vector field ξ is a regular vector field. If in addition, g is an Einstein metric, this is a Sasaki-Einstein manifold.

Remark 6.4. As consequences of Definition 6.3, we have $\phi\xi = 0$, $\eta \circ \phi = 0$, and $\nabla_X \xi = -\phi X$, in particular, $\nabla_\xi \xi = 0$. Moreover, ξ is a Killing vector field. For details, see, e.g. [Bla10].

Remark 6.5. Let us recall one more definition of Sasaki manifold. (M^n, g) is a Sasaki manifold if there exists a Killing vector field ξ of unit length on M^n so that the Riemann curvature satisfies the condition

$$(6.1) \quad R_{X\xi}Y = -g(\xi, Y)X + g(X, Y)\xi,$$

for any pair of vector fields X and Y on M^n . Then from (6.1), we can easily see that on a Sasaki-Einstein manifold (M^n, g) of dimension n , $\text{Ric}_g = (n - 1)g$.

The relationship between real Killing spinors and Sasaki-Einstein condition is revealed in [Bär93].

Theorem 6.6. (C. Bär) *A complete simply-connected Sasaki-Einstein manifold of dimension n with Einstein constant $n - 1$ carries at least 2 linearly independent real Killing spinors with distinct Killing constants equal $\frac{1}{2}$ and $-\frac{1}{2}$ for $n \equiv 3 \pmod{4}$, and to the same Killing number equals $\frac{1}{2}$ for $n \equiv 1 \pmod{4}$, respectively.*

Conversely, a complete Riemannian spin manifold with such spinors in these dimensions is Sasaki-Einstein.

Remark 6.7. *C. Bär also proved that a complete Riemannian spin manifold of even dimension n , $n \neq 6$, with a real Killing spinor is isometric to a standard sphere. Thus, a complete Riemannian spin manifold of even dimension n , $n \neq 6$, with a real Killing spinor is strictly stable.*

Remark 6.8. *In the first part of Theorem 6.6, we need at least two linearly independent real Killing spinor in order to have a Sasaki-Einstein structure. Actually, on a complete Riemannian spin manifold of odd dimension, except 7, existence of one Killing spinor automatically implies the existence of the second one that we need in Theorem 6.6. In dimension 7, we do have a complete Riemannian spin manifold with a single linearly independent Killing spinor: Jensen's sphere, which is an unstable Einstein manifold as mentioned in Introduction. More details can be found in [ADP83], [Bär93], and [Spa11].*

Now let us recall how to construct a typical regular Sasaki manifold in [Bla10]. Let (B^{2p}, G, J) be a Kähler manifold of real dimension $2p$, with the Kähler form $\Omega = G(\cdot, J\cdot)$, where G is a Riemannian metric and J is an almost complex structure. Then let $\pi : M^{2p+1} \rightarrow B^{2p}$ be a principal S^1 -bundle with a connection η with the curvature form $d\eta = 2\pi^*\Omega$. Let ξ be a vertical vector field on M^{2p+1} , generated by S^1 -action, such that $\eta(\xi) = 1$, and \tilde{X} denotes the horizontal lift of X with respect to the connection η for a vector field X on B^{2p} . We set

$$(6.2) \quad \phi X = \widetilde{J\pi_*X},$$

and

$$(6.3) \quad g(X, Y) = G(\pi_*X, \pi_*Y) + \eta(X)\eta(Y),$$

for vector fields X and Y on M^{2p+1} . Then $(M^{2p+1}, g, \phi, \eta, \xi)$ is a regular Sasaki manifold.

Conversely, any regular Sasaki manifold can be obtained in this way, see, e.g. Theorem 3.9 and Example 6.7.2 in [Bla10]. Moreover, if (M^{2p+1}, g) is Sasaki-Einstein with Einstein constant $2p$, then (B^{2p}, G, J) is Kähler-Einstein with Einstein constant $2p + 2$. We will check this fact in the following.

We fix some notations before carrying on calculations. ∇^g and ∇^G denote the Levi-Civita connections on (M^{2p+1}, g) and on (B^{2p}, G) , respectively. R^g and Ric^g , and R^G and Ric^G denote Riemann and Ricci curvatures on (M^{2p+1}, g) and on (B^{2p}, G) , respectively. In the rest of this section, we use X, Y, Z, W, \dots to denote vector fields on B^{2p} , and we use $\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W}, \dots$ to denote their horizontal lift to M^{2p+1} with respect to the connection η . And we choose and fix a local orthonormal frame $\{X_1, X_2, \dots, X_{2p}\}$ of TB . Then $\{\tilde{X}_1, \tilde{X}_2, \dots, \tilde{X}_{2p}, \xi\}$ is a local orthonormal frame of TM . We use ∇_i^g to denote $\nabla_{\tilde{X}_i}^g$, and ∇_i^G to denote $\nabla_{X_i}^G$.

Lemma 6.9. *On a regular Sasaki manifold $(M^{2p+1}, g, \phi, \eta, \xi)$ constructed above. We have*

$$(6.4) \quad [\xi, \tilde{X}] = \mathcal{L}_\xi \tilde{X} = 0,$$

$$(6.5) \quad \nabla_{\widetilde{X}}^g \widetilde{Y} = \widetilde{\nabla_X^G Y} - \Omega(X, Y)\xi,$$

$$(6.6) \quad \nabla_{\xi}^g \widetilde{X} = \nabla_{\widetilde{X}}^g \xi = -\phi \widetilde{X},$$

$$(6.7) \quad \nabla_{\xi}^g \xi = 0.$$

Proof. The equality (6.4) follows from the fact that the horizontal distribution is S^1 invariant and ξ is generated by the S^1 -action. Then the rest properties for covariant derivatives follow from properties in Remark 6.4, (6.4), and the fundamental equations of a submersion in [One66] (also see [Bes87] for the equations). \square

Let $h \in C^\infty(S^2(B))$, and then $\tilde{h} = \pi^*h \in C^\infty(S^2(M))$. Then by Lemma 6.9 and straightforward calculations, we obtain a relationship between $(\nabla^g)^*\nabla^g\tilde{h}$ and $(\nabla^G)^*\nabla^Gh$.

Lemma 6.10.

$$(6.8) \quad (\nabla_k^g \nabla_k^g \tilde{h})_{ij} = (\pi^*(\nabla_k^G \nabla_k^G h))_{ij} - 2\tilde{h}_{ij},$$

$$(6.9) \quad (\nabla_{\widetilde{\nabla_k^g X_k}}^g \tilde{h})_{ij} = (\pi^*(\nabla_{\nabla_k^G X_k}^G h))_{ij},$$

$$(6.10) \quad (\nabla_{\xi}^g \nabla_{\xi}^g \tilde{h})_{ij} = -2\tilde{h}_{ij} + 2\tilde{h}(\phi \widetilde{X}_i, \phi \widetilde{X}_j),$$

and therefore,

$$(6.11) \quad ((\nabla^g)^*\nabla^g\tilde{h})_{ij} = (\pi^*((\nabla^G)^*\nabla^G h))_{ij} + 4\tilde{h}_{ij} - 2\tilde{h}(\phi \widetilde{X}_i, \phi \widetilde{X}_j),$$

for all $1 \leq i, j \leq 2p$, where we take summation for the repeated index k through 1 to $2p$.

Because $\pi : M^{2p+1} \rightarrow B^{2p}$ is a Riemannian submersion, by the fundamental equation in [One66] and also in Theorem 9.26 in [Bes87], we have the following relationship between curvature tensors on M^{2p+1} and ones on B^{2p} .

Lemma 6.11.

$$(6.12) \quad \begin{aligned} R^g(\widetilde{X}, \widetilde{Y}, \widetilde{Z}, \widetilde{W}) = & (\pi^*R^G)(X, Y, Z, W) - 2(\pi^*\Omega)(\widetilde{X}, \widetilde{Y})(\pi^*\Omega)(\widetilde{Z}, \widetilde{W}) \\ & - (\pi^*\Omega)(\widetilde{X}, \widetilde{Z})(\pi^*\Omega)(\widetilde{Y}, \widetilde{W}) + (\pi^*\Omega)(\widetilde{X}, \widetilde{W})(\pi^*\Omega)(\widetilde{Y}, \widetilde{Z}), \end{aligned}$$

$$(6.13) \quad R^g(\widetilde{X}, \xi, \widetilde{Y}, \xi) = g(\widetilde{X}, \widetilde{Y}),$$

and therefore,

$$(6.14) \quad Ric^g(\widetilde{X}, \widetilde{Y}) = (\pi^*Ric^G)(\widetilde{X}, \widetilde{Y}) - 2g(\widetilde{X}, \widetilde{Y}).$$

From (6.14), we can see that if g is Einstein with Einstein constant k then G is also Einstein with Einstein constant $k + 2$. Moreover, the above relations between curvatures directly imply a relation between $\mathring{R}^g\tilde{h}$ and \mathring{R}^Gh .

Lemma 6.12.

$$(6.15) \quad (\mathring{R}^g \tilde{h})_{ij} = (\pi^*(\mathring{R}^G h))_{ij} - 3\tilde{h}(\phi \widetilde{X}_i, \phi \widetilde{X}_j) - (\pi^* \Omega)(\widetilde{X}_i, \widetilde{X}_j) \sum_{k=1}^{2p} \tilde{h}(\widetilde{X}_k, \phi \widetilde{X}_k),$$

for all $1 \leq i, j \leq 2p$.

Proposition 6.13.

$$(6.16) \quad \langle (\nabla^g)^* \nabla^g \tilde{h} - 2\mathring{R}^g \tilde{h}, \tilde{h} \rangle = (\langle (\nabla^G)^* \nabla^G h - 2\mathring{R}^g h, h \rangle + 4\langle h, h \rangle + 4\langle h \circ J, h \rangle) \circ \pi.$$

Therefore,

$$(6.17) \quad \begin{aligned} & \int_M \langle (\nabla^g)^* \nabla^g \tilde{h} - 2\mathring{R}^g \tilde{h}, \tilde{h} \rangle d\text{vol}_g \\ &= \int_B (\langle (\nabla^G)^* \nabla^G h - 2\mathring{R}^g h, h \rangle + 4\langle h, h \rangle + 4\langle h \circ J, h \rangle) d\text{vol}_G. \end{aligned}$$

Proof. By Lemma 6.10 and Lemma 6.12, we directly have

$$(6.18) \quad \begin{aligned} \langle (\nabla^g)^* \nabla^g \tilde{h} - 2\mathring{R}^g \tilde{h}, \tilde{h} \rangle &= (\langle (\nabla^G)^* \nabla^G h - 2\mathring{R}^g h, h \rangle + 4\langle h, h \rangle \\ &\quad + 4\langle h(J \cdot, J \cdot), h \rangle + 2(\text{tr}_G(h(J \cdot, \cdot)))^2) \circ \pi. \end{aligned}$$

Then it suffices to show that $\text{tr}_G(h(J \cdot, \cdot)) = 0$. Because (B^{2p}, G, J) is a Kähler manifold, we can choose a local orthonormal frame of TB in the form of $\{X_1, \dots, X_p, JX_1, \dots, JX_p\}$. Then

$$\text{tr}_G(h(J \cdot, \cdot)) = \sum_{i=1}^p h(JX_i, X_i) + \sum_{j=1}^p h(J^2 X_j, JX_j) = 0,$$

by using $J^2 = -id$ and the symmetry of h . \square

We choose a local orthonormal frame $\{X_1, \dots, X_p, JX_1, \dots, JX_p\}$ of TB as in the proof of Proposition 6.13, and set

$$h(X_i, X_j) = h_{ij}, \quad h(X_i, JX_j) = h_{i\bar{j}}, \quad h(JX_i, X_j) = h_{\bar{i}j}, \quad h(JX_i, JX_j) = h_{\bar{i}\bar{j}},$$

for all $1 \leq i, j \leq p$.

Then we have

$$(6.19) \quad \langle h, h \rangle = \sum_{i,j=1}^p (h_{ij}h_{ij} + h_{i\bar{j}}h_{i\bar{j}} + h_{\bar{i}j}h_{\bar{i}j} + h_{\bar{i}\bar{j}}h_{\bar{i}\bar{j}}),$$

$$(6.20) \quad \langle h \circ J, h \rangle = \sum_{i,j=1}^p 2(h_{ij}h_{\bar{i}\bar{j}} - h_{i\bar{j}}h_{\bar{i}j}) \leq \langle h, h \rangle.$$

For any $h \in C^\infty(S^2(B))$, by doing directly calculations, we have that $\text{tr}_g \tilde{h} = \text{tr}_G h$, $(\delta_g \tilde{h})(\widetilde{X}) = (\delta_G h)(X)$, and $(\delta_g \tilde{h})(\xi) = -\text{tr}_G(h(J \cdot, \cdot)) = 0$. Consequently, if h is traceless and transverse, then so is \tilde{h} .

Corollary 6.14. *If there exists a traceless transverse symmetric 2-tensor $h \in C^\infty(S^2B)$ such that $\int_B \langle (\nabla^G)^* \nabla^G h - 2\mathring{R}^g h, h \rangle d\text{vol}_G \leq -8 \int_B \langle h, h \rangle d\text{vol}_G$, then (M^{2p+1}, g) is unstable.*

Proof. Proposition 6.13 and the inequality (6.20) directly imply the conclusion. \square

Corollary 6.15. *If the base space (B^{2p}, G) of a regular Sasaki-Einstein manifold (M^{2p+1}, g) is the Riemannian product of Kähler-Einstein manifolds $(B_1^{2p_1}, G_1)$ and $(B_2^{2p_2}, G_2)$, where $p_1 + p_2 = p$, then (M^{2p+1}, g) is unstable.*

Proof. Set $h = \frac{G_1}{2p_1} - \frac{G_2}{2p_2}$. h is a traceless transverse symmetric 2-tensor and is an unstable direction of $(B^{2p}, G) = (B_1^{2p_1}, G_1) \times (B_2^{2p_2}, G_2)$. Let us recall

$$(6.21) \quad \text{Ric}_g = (2p_1 + 2p_2)g,$$

$$(6.22) \quad \text{Ric}_G = (2p_1 + 2p_2 + 2)G.$$

Then we have

$$(6.23) \quad \langle (\nabla^G)^* \nabla^G h - 2\mathring{R}^G h, h \rangle = -2 \frac{R_{G_1}}{4p_1^2} - 2 \frac{R_{G_2}}{4p_2^2} = -2(p_1 + p_2 + 1) \left(\frac{1}{p_1} + \frac{1}{p_2} \right).$$

Moreover,

$$(6.24) \quad \langle h, h \rangle = \langle h \circ J, h \rangle = \frac{1}{2p_1} + \frac{1}{2p_2}.$$

Thus, by Proposition 6.13, we have

$$(6.25) \quad \langle (\nabla^g)^* \nabla^g \tilde{h} - 2\mathring{R}^g \tilde{h}, \tilde{h} \rangle = -2(p_1 + p_2 - 1) \left(\frac{1}{p_1} + \frac{1}{p_2} \right) < 0,$$

if both $p_1 \geq 1$ and $p_2 \geq 1$. \square

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SANTA BARBARA, SANTA BARBARA, CA 93106, USA

E-mail address: `cwang@math.ucsb.edu`